MATRICES AND TRANSFORMATIONS

Operations on Matrices

The Concept of a Matrix

Explain the concept of a matrix

Definition:

A matrix is an array or an Orderly arrangement of objects in rows and columns.

Each object in the matrix is called an element (entity).

Consider the following table showing the number of students in each stream in each form.

Form	**I**	**II**	**III**	**IV**
Stream A	38	35	40	28
Stream B	36	40	34	39
Stream C	40	37	36	35

From the above table, if we enclose the numbers in brackets without changing their arrangement, then a matrix is farmed, this can be done by removing the headings and the bracket enclosing the numbers (elements) and given a name (normally a capital letter).

Nowthe above information can be presented in a matrix form as

$$A = \begin{pmatrix} 38 & 35 & 40 & 28 \\ 36 & 40 & 34 & 39 \\ 40 & 37 & 36 & 35 \end{pmatrix}$$

Any matrix has rows and columns but sometimes you may find a matrix with only row without Colum or only column without row.

In the matrix A above, the numbers 38, 36 an 40 form the first column and 38, 35, 40 and 28 form the first row.

Matrix A above has three (3) rows and four (4) columns.

In the matrix A, 34 is the element (entity) in the second row and third column while 28 lies in the first row and fourth column. The plural form of matrix is matrices.

Normallymatrices are named by capital letters and their elements by small letters which represent real numbers.

e.g. $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix.

B is a matrix containing the elements a, b, c, and d.

$C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is also a matrix which contains elements 1, 2, 3, and 4.

Order of a matrix (size of matrix)

The order of a matrix or size of a matrix is given by the number of its rows and the number of its columns.

So if A has m rows and n columns, then the order of matrix is m x n.

It is important to note that the order of any matrix is given by stating the number of its rows first and then the number of its columns.

For example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2 x 2 matrix or the order of matrix A is 2x2, and

$B = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ is a 2x3 matrix while $C = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ is a 3 x 2 matrix.

NB: $m \times n \neq n \times m$ since an m x n is a matrix with m rows and n columns while n x m is a matrix with n rows and m columns.

Types of matrices:

The following are the common types of matrices:-

(a) Zero matrixes; A zero is the matrix whose elements are all zeros.

e.g.

$A=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $B=\begin{pmatrix} 0 & 0 \end{pmatrix}$ $C=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $D=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) Square matrix:

Is a matrix whose number of rows is equal to the number of columns.

For example

$A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ or $B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$

(c) Identity Matrix: Is the square matrix whose elements in the leading diagonal are ones and zeros elsewhere.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) Column matrix: Is the matrix with only one column.

e.g. $A=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or $B=\begin{pmatrix} 1 \\ 6 \\ 9 \end{pmatrix}$

(e) Row matrix: This is a matrix with only one row.

e.g. $D=\begin{pmatrix} 1 & 10 \end{pmatrix}$ or $E=\begin{pmatrix} 3 & -2 & 17 \end{pmatrix}$

Matrices of order up to 2 X 2

Add matrices of order up to 2 X 2

When adding or subtracting one matrix from another, the corresponding elements (entities) are /added or subtracted respectively.

This being the case, we can only perform addition and subtraction of matrices with the same orders.

Example 1

Given that

A= $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and B= $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, find A+B

***Solution*;** A+B = $\begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & c+f \\ b+g & d+h \end{pmatrix}$

Matrices of order up to 2 X 2

Subtract matrices of order up to 2 X 2

Example 2

Given that

A= $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and B= $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, find A-B

Solution; A- B = $\begin{pmatrix} a & c \\ b & d \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a-e & c-f \\ b-g & d-h \end{pmatrix}$

Example 3

Solve for x, y and z in the following matrix equation;

$$\begin{pmatrix} x & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & z \\ 0 & y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

***Solution*;**

$$\begin{pmatrix} x & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & z \\ 0 & y \end{pmatrix} = \begin{pmatrix} x-3 & 2-z \\ 1-0 & 0-y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

Now x-3=2, 2-z = 0 and 0-y= 3

So x= 5, y= -3 and z = 2

Exercise 1

Determine the order of each of the following matrices;

(a) $E = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ (b) $D = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(c) $F = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ (d) $G = \begin{pmatrix} 9 \end{pmatrix}$

2. Given that

$A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$ Find (a) A+B (b) B-A

3. Given that

$A = \begin{pmatrix} 2 & 4 \\ 6 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 8 & 12 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & -1 \\ 3 & 9 \end{pmatrix}$ find;

(a) (A+B) + C

(b) A-B+C

(c) C-A- B

4. A house wife makes the following purchases during one week: Monday 2kg of meat and loaf of bread Wednesday, 1kg of meat and Saturday, 1kg of meat and one loaf of bread. The prices are 6000/= per kg of meat and 500/= per loaf of bread on each purchasing day

a. Write a 3x2 matrix of the quantities of items purchased over the three days .

b. Write a 2x1 column matrix of the unit prices of meat and bread.

5. Solve for x, y and z in the equation

$$\begin{pmatrix} x & 5 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 9 & 8 \\ y & z \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 8 & 4 \end{pmatrix}$$

Additive identity matrix.

If M is any square matrix, that is a matrix with order mxm or nxn and Z is another matrix with the same order as m such that

M+ Z= Z+M = M then Z is the additive identity matrix.

The 2x2 additive identity matrix is $Z=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The additive inverse of a matrix.

If A and B are any matrices with the same order such that A+B = Z, then it means that either A is an additive inverse of B or B is an additive inverse of A that is B=-A or A= -B

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B=\begin{pmatrix} e & g \\ h & f \end{pmatrix}$ and $Z=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then A+ B = Z implies B= Z-A

$$=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = -A, \text{ so } B=-A$$

Example 4

Find the additive inverse of A,

if A= $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

Solution: The additive inverse of A is –A= - $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

∴The additive inverse of A is $\begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix}$

Example 5

Find the additive identity of B if B is a 3×3 matrix.

Solution:

The additive identity of any n×n matrix is the n×n zero matrix.

So Z= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

A Matrix of Order 2 X 2 by a Scalar

Multiply a matrix of order 2 X 2 by a scalar

A matrix can be multiplied by a constant number (scalar) or by another matrix.

Scalar multiplication of matrices:

Rule: If A is a matrix with elements say a, b, c and d, or

A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and t is any real number, then

$$tA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix}$$

Example 6

Given that

$A = \begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix}$

Find (a) 2A (b) – 5A

Solution;

(a) $A = \begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix}$, 2A=?

$2A = 2\begin{bmatrix} 7 & 8 \\ 5 & 9 \end{bmatrix}$

$= \begin{pmatrix} 2 \times 7 & 2 \times 8 \\ 2 \times 5 & 2 \times 9 \end{pmatrix}$

$= \begin{pmatrix} 14 & 16 \\ 10 & 18 \end{pmatrix}$

$\therefore 2A = \begin{pmatrix} 14 & 16 \\ 10 & 18 \end{pmatrix}$

(b) -5A =?

$-5A = -5 \begin{pmatrix} 7 & 8 \\ 5 & 9 \end{pmatrix}$

$= \begin{pmatrix} -5 \times 7 & -5 \times 8 \\ -5 \times 5 & -5 \times 9 \end{pmatrix}$

$= \begin{pmatrix} -35 & -40 \\ -25 & -45 \end{pmatrix}$

$\therefore -5A = \begin{pmatrix} -35 & -40 \\ -25 & -40 \end{pmatrix}$

Example 7

Given,

$$B = \begin{pmatrix} 2 & 4 \\ 5 & 14 \end{pmatrix}, \text{ Find } B+B+B.$$

Solution;

$$B+B+B = 3B$$

$$3B = 3\begin{pmatrix} 2 & 4 \\ 5 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 2 & 3 \times 4 \\ 3 \times 5 & 3 \times 14 \end{pmatrix}$$

$$3B = \begin{pmatrix} 6 & 12 \\ 15 & 42 \end{pmatrix}$$

$$\therefore B+B+B = \begin{pmatrix} 6 & 12 \\ 15 & 42 \end{pmatrix}$$

Two Matrices of order up to 2 X 2

Multiply two matrices of order up to 2 X 2

Multiplication of Matrix by another matrix:

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} e & f \\ g & \mathrm{h} \end{pmatrix}$$

AB is the product of matrices A and B while BA is the product of matrix B and A.

$$\text{So } AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & \mathrm{h} \end{pmatrix}$$

In AB, matrix A is called a pre-multiplier because it comes first while matrix B is called the post multiplier because it comes after matrix A.

Rules of finding the product of matrices;

1. The pre –multiplier matrix is divided row wise, that is it is divided according to its rows.

2. The post multiplier is divided according to its columns.

3. Multiplication is done by taking an element from the row and multiplied by an element from the column.

4. In rule (iii) above, the left most element of the row is multiplied by the top most element of the column and the right most element from the row is multiplied by the bottom most element of the column and their sums are taken:

Now if A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and B = $\begin{pmatrix} e & f \\ g & \mathrm{h} \end{pmatrix}$

Then AXB = $\begin{pmatrix} a & b \\ \hline c & d \end{pmatrix} \left(\begin{array}{c|c} e & f \\ g & \mathrm{h} \end{array}\right)$

$$A\times B= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Similarly BA = $\begin{bmatrix} e & f \\ \hline g & \mathrm{h} \end{bmatrix} \left[\begin{array}{c|c} e & f \\ g & \mathrm{h} \end{array}\right]$

$$= \begin{pmatrix} ea + & fc & eb + & fd \\ ga + & \mathrm{h}c & gb + & hd \end{pmatrix}$$

Therefore it can be concluded that matrix by matrix multiplication is only possible if the number of columns in the pre-multiplier is equal to the number of rows in the post multiplier.

Example 8

Given That;

A= $\begin{bmatrix} 9 & 7 \\ 8 & 6 \end{bmatrix}$and

B = $\begin{bmatrix} 6 & -1 \\ -2 & 5 \end{bmatrix}$ find,

(a) AB (b) BA

Solution

(a) AB = $\begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix} \times \begin{pmatrix} 6 & -1 \\ -2 & 5 \end{pmatrix}$

= $\begin{bmatrix} 9 & 7 \\ 8 & 6 \end{bmatrix} \times \begin{bmatrix} 6 & -1 \\ -2 & 5 \end{bmatrix}$

$= \begin{pmatrix} 9 \times 6 + 7 \times (-2) & 9 \times (-1) + 7 \times 5 \\ 8 \times 6 + 6 \times (-2) & 8 \times (-1) + 6 \times 5 \end{pmatrix}$

$= \begin{pmatrix} 54 - 14 & -9 + 35 \\ 48 - 12 & -8 + 30 \end{pmatrix}$

$\therefore$ AB = $\begin{pmatrix} 40 & 26 \\ 36 & 22 \end{pmatrix}$

(b) BA = $\begin{pmatrix} 6 & -1 \\ -2 & 5 \end{pmatrix} \times \begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix}$

$\begin{bmatrix} 6 \times 9 + (-1) \times 8 & 6 \times 7 + (-1) \times 6 \\ (-2) \times 9 + 5 \times 8 & (-2) \times 7 + 5 \times 6 \end{bmatrix}$

$= \begin{pmatrix} 54 - 8 & 42 - 6 \\ -18 + 40 & -14 + 30 \end{pmatrix}$

$= \begin{pmatrix} 46 & 36 \\ 22 & 16 \end{pmatrix}$

$\therefore$BA = $\begin{pmatrix} 46 & 36 \\ 22 & 16 \end{pmatrix}$

From the above example it can be noted that AB≠BA, therefore matrix by matrix multiplication does not obey commutative property except when the multiplication involves and identity matrix i.e. AI=IA=A

Example 9

Let,

$P=\begin{pmatrix}4 & 2\\ 3 & 1\end{pmatrix}$ and $Q=\begin{pmatrix}5\\ 6\end{pmatrix}$ Find PQ

Solution:

$$PQ=\begin{pmatrix}4 & 2\\ 3 & 1\end{pmatrix}\times\begin{pmatrix}5\\ 6\end{pmatrix}=\begin{pmatrix}4\times 5+2\times 6\\ 3\times 5+1\times 6\end{pmatrix}$$

$$PQ=\begin{pmatrix}20+12\\ 15+6\end{pmatrix}=\begin{pmatrix}32\\ 21\end{pmatrix}$$

$$\therefore PQ=\begin{pmatrix}32\\ 21\end{pmatrix}$$

Example 10

Find C×D if

$$C=\begin{pmatrix} 2 & 4 & 6 \\ 8 & 1 & 0 \\ 3 & 5 & 7 \end{pmatrix} \text{ and } D=\begin{pmatrix} 3 & 4 \\ 10 & 6 \\ 1 & -2 \end{pmatrix}$$

Solution:

$$C=\begin{pmatrix} 2 & 4 & 6 \\ 8 & 1 & 0 \\ 3 & 5 & 7 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 10 & 6 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times3+4\times10+6\times1 & 2\times4+4\times6+6\times(-2) \\ 8\times3+1\times10+0\times1 & 8\times4+1\times6+0\times(-2) \\ 3\times3+5\times10+7\times1 & 3\times4+5\times6+7\times(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 6+40+6 & 8+24-12 \\ 24+10+0 & 32+6-0 \\ 9+50+7 & 12+30-14 \end{pmatrix}$$

$$\therefore\ CD = \begin{pmatrix} 52 & 20 \\ 34 & 38 \\ 66 & 28 \end{pmatrix}$$

Product of a matrix and an identity matrix:

If A is any square matrix and I is an identity matrix with the same order as A, then AI=IA=A

Example 11

Given;

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \text{and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find (a) AI (b) IA

Solution:

$$AI = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 3 \times 0 & 1 \times 0 + 3 \times 1 \\ 4 \times 1 + 5 \times 0 & 4 \times 0 + 5 \times 1 \end{pmatrix}$$

$$AI = \begin{pmatrix} 1 + 0 & 0 + 3 \\ 4 + 0 & 0 + 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$$

$$\therefore AI = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} = A$$

$$\text{(b) } IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$$

$$IA = \begin{pmatrix} 1 \times 1 + 0 \times 4 & 1 \times 3 + 0 \times 5 \\ 0 \times 1 + 1 \times 4 & 0 \times 3 + 1 \times 5 \end{pmatrix}$$

$$\therefore IA = \begin{pmatrix} 1 + 0 & 3 + 0 \\ 0 + 4 & 0 + 5 \end{pmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = A$$

Exercise 2

1. Given that A= (3 4) and

$B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ find (a) AB (b) BA

2. If,

$A = \begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}$ and

$B = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$ find (a) BA (b) AB

3. Using the matrices

$A = \begin{pmatrix} 9 & 7 \\ 8 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$, find

(a) 3A + 2(AC)
(b) A^2B (c) 3B-I where I is an identity matrix.

4.Find the values of x and y if

$$\begin{pmatrix} x & 1 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ y & 4 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 17 & 14 \end{pmatrix}$$

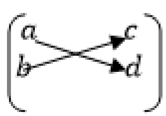
Inverse of a Matrix

The Determinant of a 2 X 2 Matrix

Calculate the determinant of a 2 X 2 matrix

Determinant of a matrix

If $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where a, b, c and d are any real numbers, then the elements (a and d) are in the leading diagonal of matrix A while the elements (b and c) are in the main diagonal.

A=


a $\rightarrow$ d = Leading diagonal

b $\rightarrow$ c= Main diagonal

Now the determinant of matrix A is then defined as the difference of the product of elements in the leading diagonal and the product of the elements in the main diagonal.

So if A= $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then determinant of A is given by ad-bc. The determinant of A is denoted by $|A|$ or det (A)

So A = $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then

$$|A| = ad-bc$$

NB: Determinants exist for square matrices only.

Example 12

Find

$|A|$ If A = $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Solution:

$|A| = 1\times 4 - 2\times 3 = 4-6 = -2$

$\therefore |A| = -2$

Example 13

Considering

If B= $\begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}$, find $|A|$

Solution: $|B|$= 1x5-0x3=5-0=5

$\therefore |B|=5$

Example 14

Find the value of x

If C = $\begin{pmatrix} 2x & 3 \\ -2 & 10 \end{pmatrix}$ having determinant 46.

Solution:

C=$\begin{pmatrix} 2x & 3 \\ -2 & 10 \end{pmatrix}$

$|C| = 2x.10 - (-2).3 = 46$

20x + 6=46

20x=46-6

20x=40

x=$\frac{40}{20}$= 2

∴The value of x is 2.

Singular and non singular matrices:

Definition:

Asingular matrix is a matrix whose determinant is zero, while non – singular matrix is the one with a non zero determinant.

For example A=$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ has determinant 1× 8 -4× $2 = 0$,

So A is a singular matrix.

Also B = $\begin{pmatrix} 6 & 1 \\ 4 & 5 \end{pmatrix}$ is a non Singular matrix because its determinant is 6x5-4x1 =30-4=26 which is not equal to zero.

Example 15

Find the value of y

If B= $\begin{pmatrix} 4 & 5 \\ 3y & 6 \end{pmatrix}$ is a Singular matrix.

Solution;

B= $\begin{pmatrix} 4 & 5 \\ 3y & 6 \end{pmatrix}$, $|B|$= 4x6-5×(3y)

$|B|$ =24-15y, but B is a singular matrix, then $|B|$ = 0

So. $|B|$ =24-15y=0

24-15y=0

24=15y

y=$\frac{15}{24}$=$\frac{5}{8}$

∴ y=$\frac{5}{8}$

The Inverse of a 2 X 2 Matrix

Find the inverse of a 2 X 2 matrix

Inverse of matrices

Definition: If A is a square matrix and B is another matrix with the same order as A, then B is the inverse of A if AB=BA=I where I is the identity matrix.

Thus AB=BA=I means either A is the inverse of B or B is the inverse of A.

For 2x2 matrix, AB =BA=I= $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So

AB=BA =I= $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The inverse of matrix A is denoted by A^{-1}

How to find A^{-1}

Let A=$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and B=$\begin{pmatrix} p & r \\ q & s \end{pmatrix}$

Where B=A^{-1}, that is B is the inverse of matrix A

$$AA^{-1}=1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

So $\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}p & r\\ q & s\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$$\begin{pmatrix}ap+cq & ar+cs\\ bp+dq & br+ds\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

From which we obtain

$ap + cq = 1$ ________________ (i)

$ar + cs = 0$__________________ (ii)

$bp + dq = 0$ _________________ (iii)

$br + ds = 1$__________________ (iv)

Since we need the unknown matrix B, we can solve for p and q by using equations (i) and (iii) and we solve for r and s using equations (ii) and (iv)

Now: $\left.\begin{matrix}ap+cq=1\\ bp+dq=0\end{matrix}\right\}$ and $\left.\begin{matrix}ar+cs=0\\ br+ds=1\end{matrix}\right\}$

By elimination method

$$\begin{cases}ap+cq=1\ldots\ldots\ldots\ldots.1\\ bp+dq=0\ldots\ldots\ldots\ldots2\end{cases}$$

$$-\begin{cases}abp+bcq=b\\ abp+adq=0\end{cases}$$

$(bc-ad)q=b$

$$q=\frac{b}{bc-ad}$$

To get p proceed as follows

$$\begin{aligned} ap + cq &= 1 \\ bp + dq &= 0 \end{aligned}$$

$$-\begin{cases} adp + dcq = d \\ cbp + dcq = 0 \end{cases}$$

(ad-bc) p=d

p= $\frac{d}{ad-bc}$

Alsoto get r and s, the same procedure must be followed:

$$\begin{matrix} b \\ a \end{matrix}\begin{cases} ar + cs = 0 \\ br + ds = 1 \end{cases}$$

$$\begin{matrix} b \\ -\ a \end{matrix}\begin{cases} abr + bcs = 0 \\ abr + bds = a \end{cases}$$

(bc-ad)s = -a

Or (ad-bc)s=a

s= $\frac{a}{ad-bc}$

And

$$d \left\{ \begin{array}{l} ar + cs = 0 \\ br + ds = 1 \end{array} \right. \quad c$$

$$- \begin{cases} adr + cds = 0 \\ bcr + cds = c \end{cases}$$

(ad-bc) r= -c

Or (bc-ad) r=c

r=$\frac{c}{bc-ad}$

Therefore p=$\frac{d}{ad-bc}$, q=$\frac{b}{\text{bc}-\text{ad}}$, r=$\frac{c}{bc-ad}$ and s=$\frac{a}{ad-bc}$.

Remember B = $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ which is the inverse of A where A=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So

$$B=\begin{pmatrix} \frac{d}{ad-bc} & \frac{c}{bc-ad} \\ \frac{b}{\text{bc}-\text{ad}} & \frac{a}{ad-bc} \end{pmatrix}$$

But bc-ad = - (ad-bc).

Now

$$B=\begin{pmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{\text{ad}-\text{bc}} & \frac{a}{ad-bc} \end{pmatrix}$$

Hence $A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$

But ad-bc = $|A|$

Then $A^{-1} = \begin{pmatrix} \frac{d}{|A|} & \frac{-c}{|A|} \\ \frac{-b}{|A|} & \frac{a}{|A|} \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ then

$\therefore A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Note that, if |A|= 0, Then

$A^{-1} = \frac{1}{|0|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ which is **Undefined**.

Example 16

Given that,

$A=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ find:

(a) A^{-1} (b) $(A^2)^{-1}$

Solution:

(a) A^{-1} =?

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$ = 1x4-2x3= -2

$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$

$\therefore A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{pmatrix}$

(b) $A^2 = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \text{x} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$= \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$

$|A^2|$= 22x7-15x10

det (A^2) = 4

$(A^2)^{-1} = \frac{1}{4}\begin{pmatrix} 22 & -15 \\ -10 & 7 \end{pmatrix}$

$= \begin{pmatrix} \frac{11}{2} & \frac{-15}{4} \\ \frac{-5}{2} & \frac{7}{4} \end{pmatrix}$

$\therefore (A^2)^{-1} = \begin{pmatrix} \frac{11}{2} & \frac{-15}{2} \\ \frac{-5}{2} & \frac{7}{4} \end{pmatrix}$

Example 17

Which of the following matrices have inverses?

(a) $B=\begin{pmatrix}-1 & 3\\ 2 & 6\end{pmatrix}$ (b) $C=\begin{pmatrix}3 & 6\\ 4 & 8\end{pmatrix}$ (c) $D=\begin{pmatrix}2 & 9\\ 4 & 18\end{pmatrix}$

Solution:

(a) $B=\begin{pmatrix}-1 & 3\\ 2 & 6\end{pmatrix}$, $|B| = -1\text{x}6-2\text{x}3$

$|B| = -12 \neq 0$

$\therefore B^{-1}$ exists

(b) $C=\begin{pmatrix}3 & 6\\ 4 & 8\end{pmatrix}$, $|C| = 3\text{x}8-4\text{x}6 = 24-24 = 0$

$|C| = 0$, so C^{-1} does not exist

(c) $D=\begin{pmatrix}2 & 9\\ 4 & 18\end{pmatrix}$, $|D| = 2\text{x}18-9\text{x}4 = 36-36 = 0$

$\therefore$ D^{-1} does not exist.

Exercise 3

1. Find the determinant of each of the following matrices.

(a) $A=\begin{pmatrix}4 & 4\\ -4 & 4\end{pmatrix}$ (b) $B=\begin{pmatrix}3 & 6\\ 9 & 15\end{pmatrix}$ (c) $C=\begin{pmatrix}x & 1\\ 3 & y\end{pmatrix}$

2. Which of the following matrices are singular matrices?

(a) $E=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ (b) $F=\begin{pmatrix}3 & 2\\ 6 & 4\end{pmatrix}$ (c) $G=\begin{pmatrix}12 & 3\\ 4 & -1\end{pmatrix}$

3. Findinverse of each of the following matrices.

(a) A= $\begin{pmatrix} 1 & 6 \\ 9 & 57 \end{pmatrix}$ (b) B = $\begin{pmatrix} x & 2 \\ 3 & y \end{pmatrix}$ Where $xy \neq 6$

2 X 2 Matrix to Solve Simultaneous Equations

Apply 2 X 2 matrix to solve simultaneous equations

Solving simultaneous equations by matrix method:

If A and B are two matrices Such that A = $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and B = $\begin{pmatrix} x \\ y \end{pmatrix}$

Then AB = $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ x $\begin{pmatrix} x \\ y \end{pmatrix}$ = $\begin{pmatrix} ax + & cy \\ bx + & dy \end{pmatrix}$

Where a, b, c, d, x and y are any real numbers.

Let AB=C where C= $\begin{pmatrix} p \\ q \end{pmatrix}$

AB=C means $\begin{pmatrix} ax + & cy \\ bx + & dy \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Now by equating the corresponding elements, the following simultaneous equations are obtained.

$$ax + \quad cy = p \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$
$$bx + \quad dy = q \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Therefore for any system of simultaneous equations, a matrix method can be used if and only if ad-bc≠0.

So, because ax+cy=p and bx+dy=q can be written, in a matrix equation form as $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ then we can Let A $=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, B= $\begin{pmatrix} x \\ y \end{pmatrix}$ and C= $\begin{pmatrix} P \\ q \end{pmatrix}$ and the equation becomes AB=C.

If A^{-1} is multiplied on each side of the equation, then

A^{-1} x AxB= A^{-1}xC

(A^{-1} A) x B= A^{-1} xC

But A^{-1} xA = I where I is an identity matrix.

Also for any matrix, K, KI=IK = K

Therefore (A^{-1}xA) xB=IxB= A^{-1}C

Then B= $A^{-1}\times$C

But B= $\begin{pmatrix} x \\ y \end{pmatrix}$, C= $\begin{pmatrix} P \\ q \end{pmatrix}$

And $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ where $|A| = ad - bc$

$$\therefore \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}\begin{pmatrix} P \\ q \end{pmatrix}}$$

Or x = $\frac{dp-cq}{ad-bc}$ and y = $\frac{aq-bp}{ad-bc}$

Example 18

By matrix method solve the following simultaneous equations:

5x+6y=11 ______________ (*i*)
7x+8y=15 ______________ (*ii*)

Solution:

The system $\begin{cases} 5x + by = 11 \_\_\_\_\_\_\_\_\_\_ (i) \\ 7x + 8y = 15 \_\_\_\_\_\_\_\_\_\_ (ii) \end{cases}$ can be written in matrix form equation as

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \text{x} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 15 \end{pmatrix}$$

Let A = $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

$$A^{-1} = \frac{1}{5\times8-6\times7}\begin{pmatrix} 8 & -6 \\ -7 & 5 \end{pmatrix}$$

$$A^{-1} = \frac{1}{40-42}\begin{pmatrix} 8 & -6 \\ -7 & 5 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 8 & -6 \\ -7 & 5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{pmatrix}$$

Multiplying A^{-1} an each side of the equation, gives,

$$\begin{pmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ \frac{7}{2} & \frac{-5}{2} \end{pmatrix} \begin{pmatrix} 11 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{x} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \times 11 \ + 3 \times 15 \\ 7 \times \frac{11}{2} \ + -5 \times \frac{15}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -44 \ + 45 \\ \frac{77}{2} - \frac{75}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 45 - 44 \\ \frac{77}{2} - \frac{75}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

∴x=1 and y=1

Example 19

Solve

$$\begin{cases} 4x + 2y = 40 \\ x + 3y = 35 \end{cases}$$ By matrix method

Solution:

The above system is equivalent to the matrix equation

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 35 \end{pmatrix}$$

Let A = $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$

$$A^{-1} = \frac{1}{4\times3-1\times2}\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{-1}{5} \\ \frac{-1}{10} & \frac{2}{5} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{3}{10} & \frac{-1}{5} \\ \frac{-1}{10} & \frac{2}{5} \end{pmatrix}$$

Multiplying A^{-1} on each side of the equation gives,

$$\begin{pmatrix} \frac{3}{10} & \frac{-1}{5} \\ \frac{-1}{10} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{-1}{5} \\ \frac{-1}{10} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} 40 \\ 35 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3\times40}{10} - \frac{1\times35}{5} \\ \frac{-1\times40}{10} + \frac{2\times35}{5} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12-7 \\ -4 \ +14 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

∴x =5 and y=10

Example 20

By using matrix method solve the following simultaneous equations:

$$3x - y = 11$$
$$x + 3y = -3$$

Solution:

We can write the above equations

As $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}$

Let A= $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$

$A^{-1} = \frac{1}{3\times3-1\times(-1)} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$

$A^{-1} = \frac{1}{10} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{-1}{10} & \frac{3}{10} \end{pmatrix}$

Multiplying A^{-1} on each side of the equation gives,

$$\begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{-1}{10} & \frac{3}{10} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{-1}{10} & \frac{3}{10} \end{pmatrix} \begin{pmatrix} 11 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3\times11}{10} - \frac{1\times3}{10} \\ \frac{-1\times11}{10} - \frac{3\times3}{10} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{30}{10} \\ \frac{-20}{10} \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$\therefore$ x = 3 and y = -2

Cramer's Rule

Cramer's rule is another method to solve the equations of the form $\begin{cases} ax \times cy = p \\ bx + dy = q \end{cases}$

By using the inverse of a matrix method we have seen that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} P \\ q \end{pmatrix}$$

Or x = $\frac{dp-cq}{ad-bc}$ and y = $\frac{aq-bp}{ad-bc}$

Where ad-bc is the determinant of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

So

$x = \frac{\begin{vmatrix} p & c \\ q & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$ and $y = \frac{\begin{vmatrix} d & p \\ b & q \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$

In $x = \frac{\begin{vmatrix} p & c \\ q & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$ the numerator is obtained by replacing the first column by the column $\begin{pmatrix} P \\ q \end{pmatrix}$ and in $y = \frac{\begin{vmatrix} d & p \\ b & q \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$, the numerator is obtained by taking the determinant of the matrix after replacing the second column by the column $\begin{pmatrix} P \\ q \end{pmatrix}$ While the denominator in both cases is the determinant of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Example 21

Find

X and y from $\begin{cases} 5x + 6y = 11 \\ 7x + 8y = 15 \end{cases}$ by using Cramer's rule.

Solution: The equation can be written as

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 15 \end{pmatrix}$$

$x = \frac{\begin{vmatrix} 11 & 6 \\ 15 & 8 \end{vmatrix}}{\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}}$ and $y = \frac{\begin{vmatrix} 5 & 11 \\ 7 & 15 \end{vmatrix}}{\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}}$

$x = \frac{11\times8-6\times15}{5\times8-7\times6} = \frac{88-90}{40-42} = \frac{-2}{-2} = 1$ and $y = \frac{5\times15-7\times11}{5\times8-7\times6} = \frac{75-77}{40-42} = \frac{-2}{-2} = 1$

$\therefore x=1$ and $y=1$

Example 22

By using Cramer's rule

, find x and y in $\begin{Bmatrix} 4x+2y=40 \\ x+3y = 35 \end{Bmatrix}$

Solution;

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 40 \\ 35 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} 40 & 2 \\ 35 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} 4 & 40 \\ 1 & 35 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix}}$$

$$x = \frac{40\times3-35\times2}{4\times3-1\times2} = \frac{120-70}{12-2} = \frac{50}{10} = 5 \text{ and } y = \frac{4\times35-40\times1}{4\times3-1\times2} = \frac{140-40}{12-2} = \frac{100}{10} = 10$$

∴x=5 and y=10

Example 23

Byusing Cramer's rule,

Solve $\begin{cases} 3x-y=11 \\ x+3y=-3 \end{cases}$

Solution:

The equations can be written as $\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}$

$$\text{So } x = \frac{\begin{vmatrix} 11 & -1 \\ -3 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} 3 & 11 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}}$$

$$x = \frac{11\times3--3\times(-1)}{3\times3-1\times(-1)} = \frac{33-3}{9+1} = \frac{30}{10} = 3 \text{ and } y = \frac{3\times(-3)-1\times11}{3\times3-7\times6} = \frac{-9-11}{9+1} = \frac{-20}{10} = -2$$

∴x=3 and y= -2

Exercise 4

1. Use the matrix method to solve the following systems of simultaneous equations.

(a) $\begin{cases} 7x - 2y = 29 \\ 7x + y = 38 \end{cases}$ (b) $\begin{cases} 3x + 2y = 12 \\ 7x - 4y = 2 \end{cases}$ (c) $\begin{cases} 3x - 2y = 7 \\ 4x + 5y = 40 \end{cases}$

(d) $\begin{cases} 4x - 6 = 3y \\ 4 + 5y = -2x \end{cases}$

Use Cramer's rule to solve the following simultaneous equation

(a) $\begin{cases} 3x + 4y = 8 \\ 2x + 3y = 13 \end{cases}$ (b) $\begin{cases} 2x - 3y = 4 \\ 2x + 3y = 6 \end{cases}$ (c) $\begin{cases} -2y + 2x = 7 \\ 4x - 5y = 2 \end{cases}$

3. Whythe system of simultaneous equations

$\begin{cases} 3x + 2y = 5 \\ 6x + 4y = 8 \end{cases}$ has no solution?

Matrices and Transformations

Definition: A transformation in a plane is a mapping which moves an object from one position to another within the plane. Figures on the plane can also be shifted from one position by a transformation.

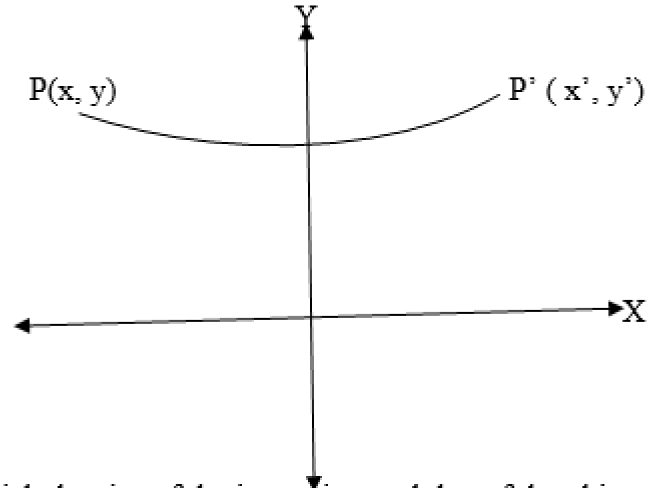
A new position after a transformation on is called the ***image***.

Examples of transformations are (i) Reflection (ii) Rotation (iii) Enlargement (iv) Translation.

Any Point P(X, Y) into $P^1(X^1,Y^1)$ by Pre-Multiplying $\begin{pmatrix} x \\ y \end{pmatrix}$ with a Transformation Matrix T

Transform any point P(X, Y) into $P^1(X^1,Y^1)$ by pre-multiplying $\begin{pmatrix} x \\ y \end{pmatrix}$ with a transformation matrix T

- Suppose a point P(x,y) in the x-y plane moves to a point P¢ (x¢,y¢) by a transformation T, We say that P is mapped to P′ by T and may be indicated as $P \xrightarrow{T} P'$



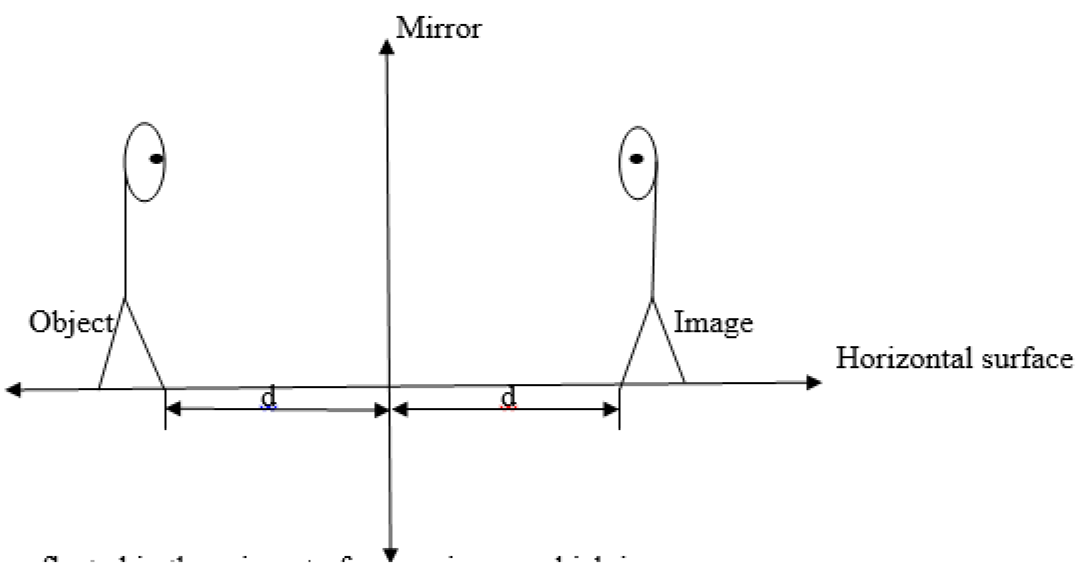
A transformation in which the size of the image is equal that of the object is called an ISOMETRIC MAPPING.

The Matrix to Reflect a Point P(X, Y) in the X-Axis

Apply the matrix to reflect a point P(X, Y) in the x-axis

Reflection;

When you look at yourself in a mirror you seem to see your body behind the mirror. Your body is in front of the mirror as your image is behind it.



An object is reflected in the mirror to form an image which is;

a. The same size as the object

b. The same distance from the mirror as the object

So reflection is an example of ISOMETRIC MAPPING.

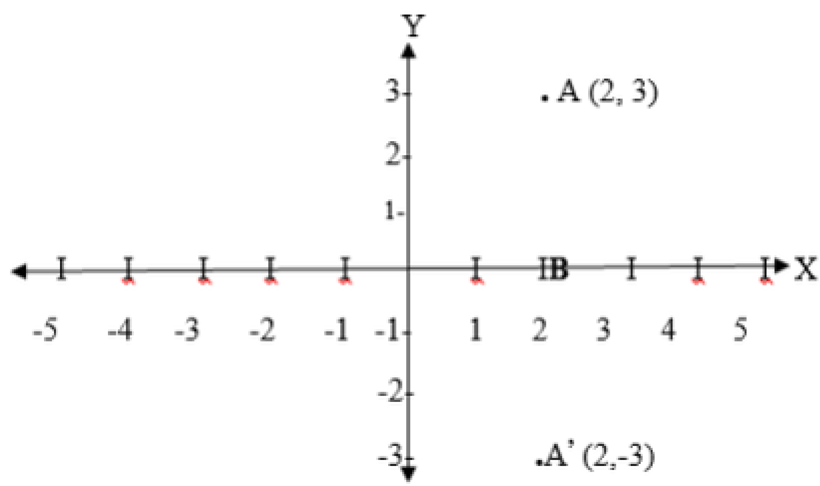
The mirror is the line of symmetry between the object and the image.

Example 24

Find the image of the point A (2,3) after reflection in the x – axes.
Solution;

Plot point A and its image A¢ such that AA¢ crosses the x – axis at B and also perpendicular to it.

For reflection AB should be the same as BA¢ i.e. AB = BA¢



From the figure, the coordinates of A ¢ are A¢ (2,-3). So the image of A (2,3) under reflection in the x-axis is A¢ (2,-3)

Normally the letter M is used to denote reflection and thus Mx means reflection in the x – axis.

So Mx(2,3) =- (2,-3).

Generally

Mx(x, y) =(x,-y) and My(x, y) = (-x, y)

Where Mx means reflection in the x – axis and My means reflection in the y-axis.

The Matrix to Reflect a Point P(X, Y) in the Y-Axis

Apply the matrix to reflect a point P(X, Y) in the Y-Axis

Example 25

Find the image of B(3,4) under reflection in the y- axis.

Solution:

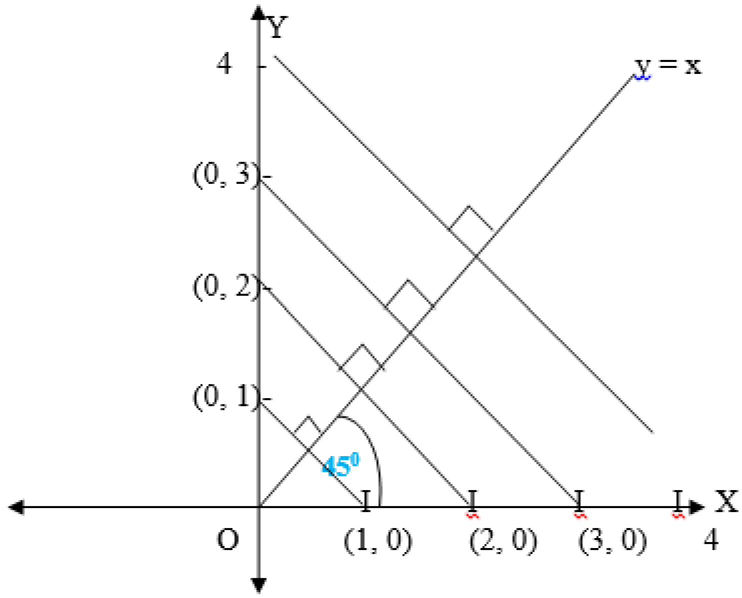
From My (x.y)= (-x,y)

My (3 ,4) =(-3,4)

Therefore the image of B(3,4) is **B'(-3,4)** .

Reflection in the line y = x.

The line y=x makes an angle 45^0 with x and y axes. It is the line of symmetry for the angle YOX formed by two axis. By using isosceles triangle properties, reflection of the point (1,0) in the line y=x will be (0,1) while the reflection of (0,2) in the line y=x will be (2, 0) it can be noticed that the coordinates are exchanging positions. Hence the reflection of the point (x,y) in the line y=x is (y,x).



Generally

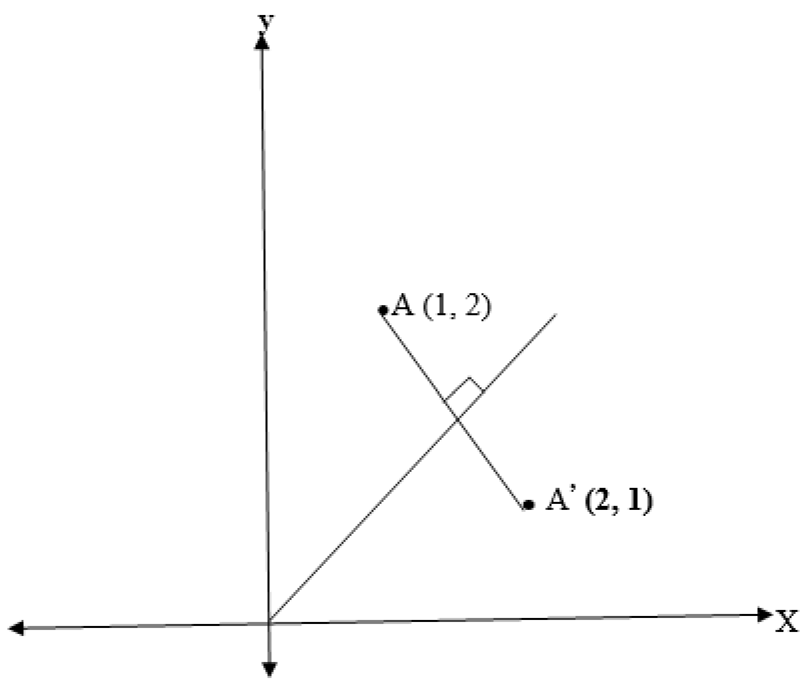
$$M_{y=x}(x, y) = (y, x)$$

Where $M_{y=x}$ means reflection in the line y=x.

Example 26

Find the image of the point A(1,2) after reflection in the line y = x . Draw a sketch.

Solution;



Therefore the image of A '(1, 2) is A′ (2, 1).

Reflection in the line y = -x

The reflection of the point B(x,y) in the line y = -x is B'(-y,-x).

i.e.

$$M_{y=-x}(x, y) = (-y,-x)$$

Where by $M_{y=-x}$ means reflection in the line y = -x.

Example 27

Find the image of B (3,4) after reflection in the line y=-x followed by another reflection in the line y=0.Draw a sketch.

Solution;

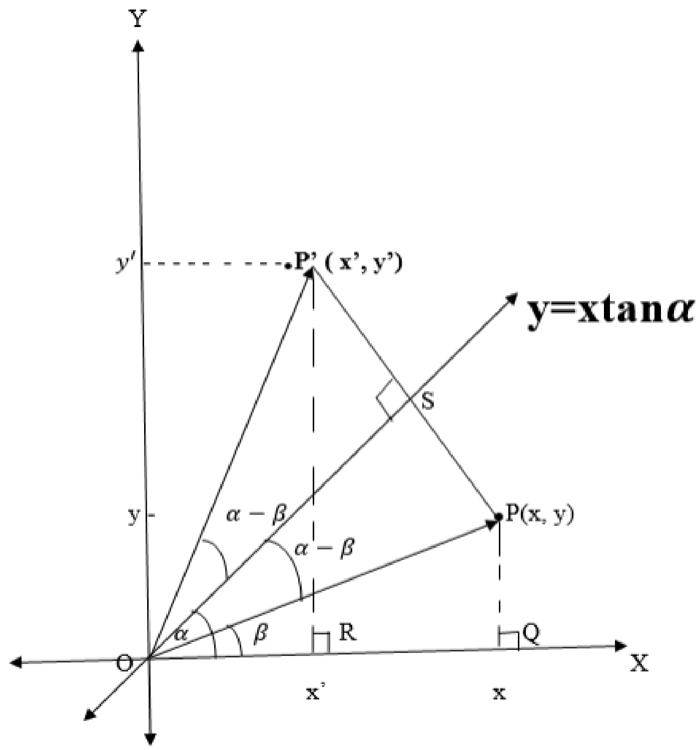
Reflection of B in the line y=-x is B'(-4,-3). The line y=0 is the x – axis. So reflection (-4,-3) in the x-axis is (-4,3)

Therefore the image of B (3,4) is **B¢(-4,3).**

***The image of a point P(x,y) when reflected in the line making an angle α with positive x-axis and passing through the origin*.**

If the line passes through the origin and makes an angle a with x – axis in the positive direction, then its equation is y= xtanα where tanαis the slope of the line.

Consider the following diagram.



From the figure above $\overrightarrow{OP}$ is inclined at β and the coordinates of P are (x, y).

$\overrightarrow{OP'}$ is the image of $\overrightarrow{OP}$ under reflection in the $\overrightarrow{OS}$.

Now by physical laws of reflection PP' is perpendicular to the line of reflection, in this case the line of reflection is OS which is given by the equation y = xtanα.

Alsop P′OS= POS= α - β and the line segment OP is equal to the line segment OP'.

Since the angle $\overrightarrow{P'R}$ and $\overrightarrow{PQ}$ are both perpendicular to the x – axis, then according to coordinate geometry the coordinates of Q are (x, 0) and that of R are (x', 0).

So OQ = x and OR = x′ while QP = y and RP′= y′.

But OPQ is a right angled triangle.

So $x = OP\, Cos\beta$ and $y = OPSin\beta$.

Again OP¢R is a right angled triangle and the angle P¢QR = a -β + a- β+ β, this is due to the fact that reflection is an isometric mapping.

Now the angle P¢OR = 2 a-β, then

$x' = OP' [\, Cos\, (\alpha\, B - \beta)]$ and $y' = OP' [Sin\, (2\alpha\text{-}\beta)]$, on expanding

$x' = OP\, '[Cos\, 2\alpha Cos\, \beta + Sin\, 2\, \alpha\ \ Sin\beta]$…………………(1)

And $y' = OP'[\, Sin\, 2\, \alpha\, Cos\beta - Sin\beta Cos\, 2\alpha]$……………… (2)

But $OP = OP,'$

Then $x' = OP\, [\, Cos2\alpha Cos\beta + Sin2\alpha Sin\beta]$

And $y' = OP\, [\, Sin2\alpha\, Cos\beta - Sin\beta Cos2\alpha\,]$

Which implies that

$X'\, (OPCos\beta)\, Cos2\alpha + (OPSin\beta)\, Sin2\alpha$ and

$y' = (OP\, Cos\beta)\, Sin2\alpha - (OPSin\beta)\, Cos2\alpha$

Remember $x = OPCos\beta$ and $y = OPSin\beta$

Therefore $x' = xCos2\alpha + ySin\, 2\alpha$

And $y' = xSin2\alpha - yCos2\alpha$

It follows therefore that if M is a reflection in the line inclined at a, then

M[x, y] = [x′, y′] where

$$\begin{cases} x' = x\cos 2\alpha + y\sin 2\alpha \quad \text{and} \\ y' = x\sin 2\alpha - y\cos 2\alpha \end{cases}$$

The above two linear equations are called transformation equations for reflection. These two equations can be written in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Where $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the position vector which is the image of $\begin{pmatrix} x \\ y \end{pmatrix}$ under reflection M,

And $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ is the matrix of reflection.

NB $\begin{pmatrix} x \\ y \end{pmatrix}$ = (x, y) and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ = (x′, y′)

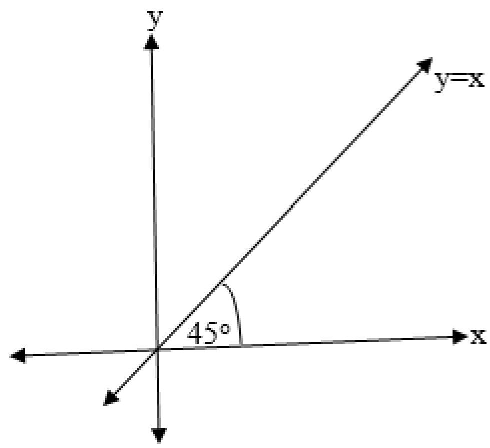
Example 28

Find the image of the point A (1, 2) after a reflection in the line y = x.

Solution:

The line y = x has a slope 1

So tan α = 1, α = 45^0



From M α = $\begin{pmatrix} Cos2\alpha & Sin2\alpha \\ Sin2\alpha & -Cos2\alpha \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$,

Then $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos90° & Sin90° \\ Sin90° & -Cos90° \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, but (x, y) = (1, 2)

So $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ = (2, 1)

Therefore (x', y') = (2, 1) which is the image of (1, 2).

Example 29

Find the image of B (3,4) after reflection in the line y = -x followed by another reflection in the line y = 0.

Solution:

The line y = -x has slope -1

So tan α = -1

$\alpha = -45^0 = 135^0$

$$M\,\alpha = \begin{bmatrix} Cos2\alpha & Sin2\alpha \\ Sin2\alpha & -Cos2\alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

But (x, y) = (3, 4)

$$M_{135}{}^{o} = \begin{bmatrix} Cos270° & Sin270° \\ Sin270° & -Cos270° \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 + -4 \\ -3 + 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

Now we need the image of (-4,-3) after being reflected in the line y = 0.

But the line y = 0 has 0 slope because it is the x – axis,

Therefore tan α = 0, $\alpha = 0^{0.}$

$$M_0{}^0 = \begin{bmatrix} Cos0° & Sin0° \\ Sin0° & -Co0° \end{bmatrix}\begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 + -0 \\ 0 + 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Hence $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Note that $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

Also it should be noted that matrix multiplication does not obey communicative property, this is to say

$M\,\alpha_1 M\,\alpha_2 = M\,\alpha_2\ M\,\alpha_1$ if and only if $\alpha_1 = \alpha_2$.

Example 30

Find the equation of the line y = 2x + 5 after being reflected in the line y = x,

Solution:

The line y = x has a slope 1

So tan a = 1 which means a = 45^0

To find the image of the line y = 2x + 5, we choose at least two points on it and find their images, then we use the image points to find the equation of the image line.

Now y = 2x + 5

x	0	1
y	5	7

The points (0,5) and (1,7) lie on the line

But M α = $\begin{bmatrix} Cos2\alpha & Sin2\alpha \\ Sin2\alpha & -Cos2\alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

So 2α=2x45^0 = 90^0

For (0, 5), $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} Cos90° & Sin90° \\ Sin90° & -Cos90° \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0+5 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

For (1, 7), $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} Cos90° & Sin90° \\ Sin90° & -Cos90° \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 0+7 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

So the image line is the line passing through (5,0) and (7,1) and it is obtained as follows;

Slope = $\frac{1-0}{7-5} = \frac{1}{2}$

Equation: $\frac{y-0}{x-5} = \frac{1}{2}$

2y = x – 5

y = $\frac{x-5}{2}$

Therefore the equation of the line y = 2x + 5 after being reflected in the line y = x is **2y = x – 5.**

Exercise 5

Self Practice.

1. Find the image of the point D (4,2) under reflection in the x – axis

2. Point Q (-4,3) is reflected in the y – axis. Find its image coordinates.

3. Reflect the point (5,4) in the line y = x

4. Find the image of the point (1,2) after a reflection in the line y = x followed by another reflection in the line y = -x.

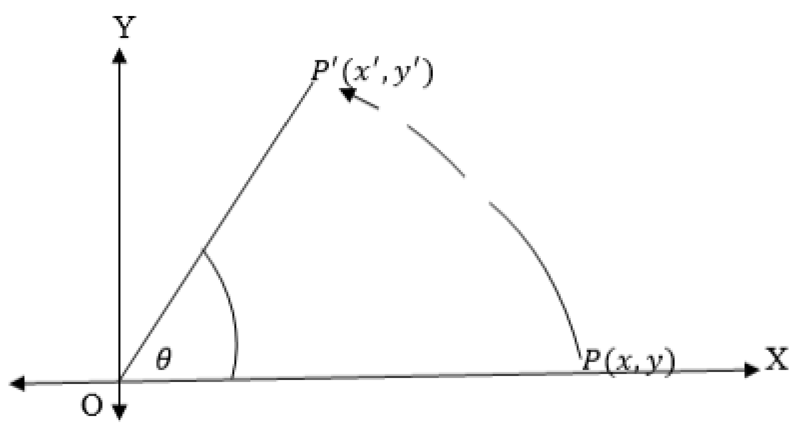
5. Find the equation of the line y = 3x -1 after being reflected in the line x + y = 0.

A Matrix Operator to Rotate any Point P(X, Y) Through 90° 180°, 270° and 360° about the Origin

Use a matrix operator to rotate any point P(X, Y) through 90° 180°, 270° and 360° about the Origin

Rotation:

Definition; A rotation is a transformation which moves a point through a given angle about a fixed point.

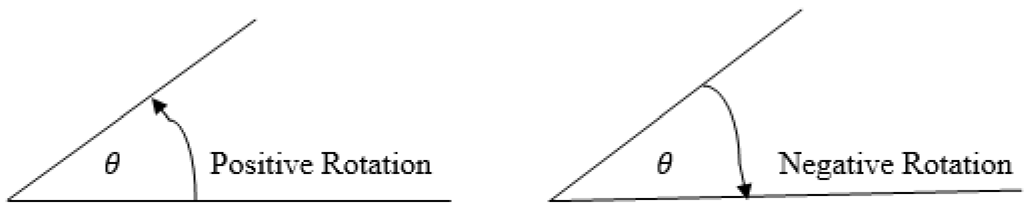


The point $P(x, y)$ is rotated through angle θ to form an image $P'(x', y')$.

Rotation is an isometric mapping and it is usually denoted by R.

Therefore Rθ means rotation of an object through an angleθ.

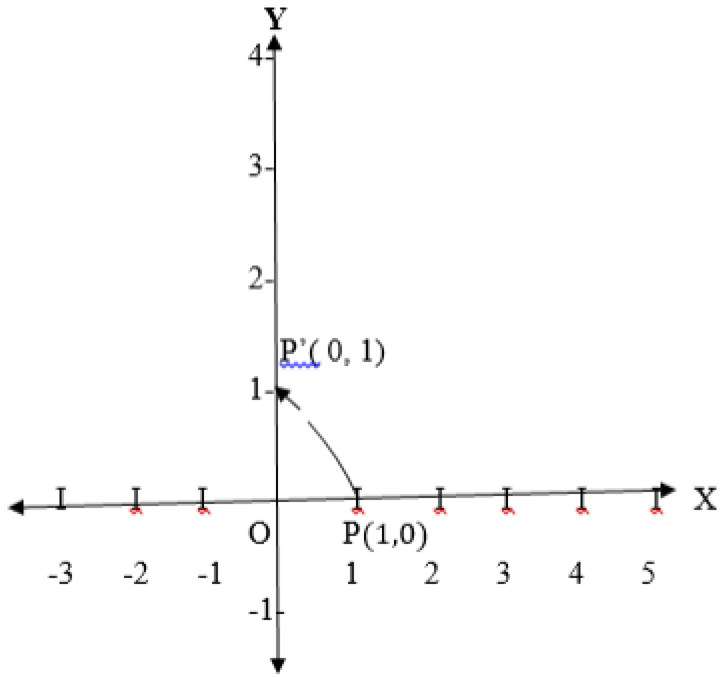
In the xy plane, whenθismeasured in the clockwise direction it is negative and when it is measured in the anticlockwise direction it is positive.



Example 31

Find the image of the point P(1,0) after a rotation through 90^0 about the origin in the anti clockwise direction.

Solution:



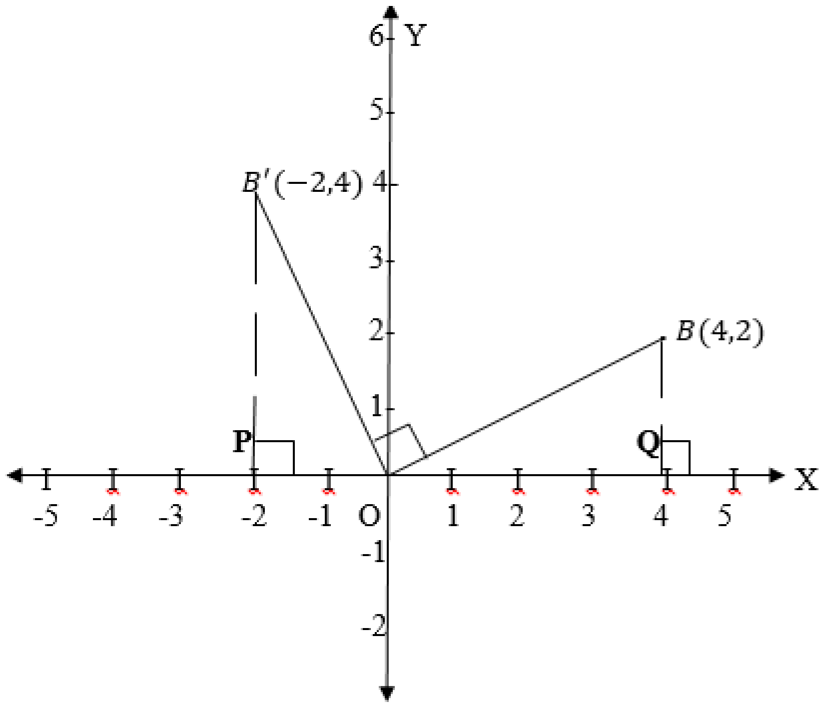
P is on the x – axis, so after rotation through 90^0 about the origin it will be on the y – axis. Since P is 1unit from O, P¢ is also 1 unit from O, the coordinates of P¢ (0,1) are P¢ (0,1). Therefore $R_{90^0}(1,0) = (0,1)$.

Example 32

Find the image of the point B (4,2) after a rotation through 90^0 about the origin in the anticlockwise direction.

Solution;

Consider the following figure,



From the figure, OB = OB′ and QBO = POB′

$\Delta POB \equiv \Delta QBO$

So $PB' = OQ$

Hence $y' = 4$ and $x' = -2$

Therefore $R_{90^0}(4, 2) = $ **(-2, 4)**

Exercise 6

Find the matrix of rotation through

a. 90^0 about the origin

b. 45^0 about the origin

c. 270^0 about the origin

Find the image of the point (1,2) under rotation through 180^0 ant –clockwise about the origin.

Find the image of the point (-2,1) under rotation through 270^0 clockwise about the origin

Find the image of (1,2) after rotation of -90^0.

Find the image of the line passing through points a (-2,3) and B(2,8) after rotation through 90^0 clockwise about the origin

General formula for rotation

Consider the following sketch,

From the figure above, $\overrightarrow{OP}$ is inclined at angle β, also $|\overrightarrow{OP}|$ is inclined angle β, also $|OP|$ because they are radii of the circle whose center is the point (0, 0).

Again OA = x, AP =y, OB = x′ and BP′ = y′

Now in the $\triangle POA$, the angle POA = β

So $x = |OP| \cos\beta$ and $y = |OP| \sin\beta$

In the $\Delta P'OB$, the angle $P'OB = \alpha+\beta$

So $\cos(\alpha+\beta) = \frac{|OB|}{|OP'|} = \frac{x'}{|OP'|}$

$x' = |OP'| \cos(\alpha+\beta)$

While $\sin(\alpha+\beta) = \frac{|P'B|}{|OP'|} = \frac{y'}{|OP'|}$

$y' = |OP'| \sin(\alpha+\beta)$.

On expanding Cos (α+β) and Sin (α+β), we get

$x' = OP'(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$ and

$y' = OP'(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$

but $OP = OP'$

$x' = OP(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$ and

$y' = OP(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$

$x' = |OP| \cos\alpha\cos\beta - |OP| \sin\alpha\sin\beta$

and $y' = |OP| \sin\alpha\cos\beta + |OP| \sin\beta\cos\alpha$

but $|OP| \cos\beta = x$ and

$|OP| \sin\beta = y$

Then $x' = x\cos\alpha - y\sin\alpha$

$y' = x\sin\alpha + y\cos\alpha$

which can be written in matrix form as;

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} Co\alpha & -Sin\alpha \\ Sin\alpha & Cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$R\alpha = \begin{bmatrix} Co\alpha & -Sin\alpha \\ Sin\alpha & Cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Where $R\alpha$ is the rotation of the point (x, y) through the angle α and $\begin{bmatrix} x' \\ y' \end{bmatrix}$ is the image of the point $\begin{bmatrix} x \\ y \end{bmatrix}$.

Example 33

Find the image of the point (1,2) under a rotation through 180^0 anticlockwise

Solution:

Given (x,y) = (1,2)

$\alpha = 180^0$

$$R\alpha = \begin{bmatrix} Cos\alpha & -Sin\alpha \\ Sin\alpha & Cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} Co180° & -Sin180° \\ Sin180° & Cos180° \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+0 \\ 0+-2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Therefore the image of (1, 2) after rotation through 180^0 anticlockwise is (-1,-2).

Example 34

Find the image of the point (5,2) under rotation of 90^0 followed by another rotation of 180^0anticlockwise.

Solution:

$(x, y) = (5, 2)$, $\alpha_1 = 90^0$ and $\alpha_2 = 180^0$

From

$$R\alpha = \begin{bmatrix} Cos\alpha & -Sin\alpha \\ Sin\alpha & Cos\alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix},$$

$$R90° = \begin{bmatrix} Cos90° & -Sin90° \\ Sin90° & Cos90° \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix},$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0-2 \\ 5+0 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

So $R90°(5,2) = (-2,5)$,

Also

$$R180° = \begin{bmatrix} Cos180° & -Sin180° \\ Sin90° & Cos180° \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix},$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 25 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0+-5 \end{bmatrix}$$

Therefore the image of (5,2) under rotation of 90^0 followed by another rotation of 180^0anticlockwise is (2,-5) .

Translation

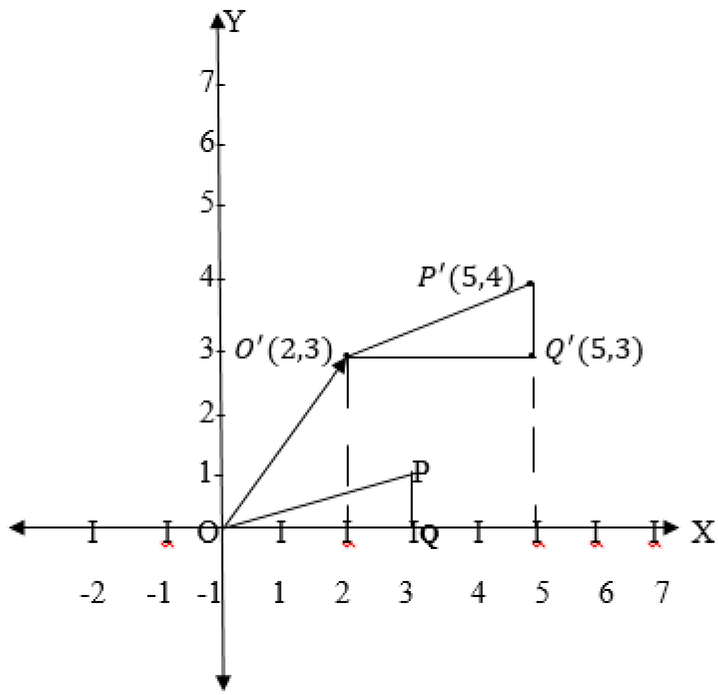
Definition: A translation is a mapping of a point P (x, y) into P' (x', y') by the Vector (a, b) such that (x', y') = (x, y) + (a, b), translation is denoted by the letter T. So T maps a point (x, y) into x', y')

Where (x', y') = (x, y) + (a, b)

Or

(x', y') = (x, y) + (a, b)

Consider the triangle OPQ whose vertices are (0,0), (3,1) and (3,0) respectively which is mapped into triangle O¢P¢Q¢ by moving it 2 units in the positive x direction and 3 units in the positive y direction



From the figure above, the image of ΔOPQ is ΔO'P'Q' whose vertices are (2,3), (5,4) and (5,3) respectively.

Note that $\overrightarrow{OQ}$= (2, 3) is called the vector of translation. Also the size of the image is equal to that of object.

Example 35

If T is a translation by the vector (4,3), find the image of (1, 2) under this translation.

Solution;

From T(x, y) = (x′,y′) where

(x′,y′) = (x, y) + (a, b)

(x′, y′) = (1, 2) + (4,3) = (5,5).

Therefore the image of (1, 2) under the translation T is (5, 5).

Example 36

A translation T maps the point (-3, 2) into (4, 3). Find where (a) T maps the origin (b) T maps the point (7, 4).

Solution:

From T(x, y) = (x', y') , then

(x', y') = (x, y) + (a, b),

where (a, b) is the translation vector,

Now $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$

Then

4 = -3+a and 3 = 2+b, solving the two equations gives a = 7 and b = 1, so **(a, b) = (7, 1)**

(a) T(0,0) = $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

Therefore T maps the point (0,0) into (7,1)

(b) T (7,4) = (7,4)+(7,1) = (14,5).

Therefore T maps the point (7,4) into (14,5).

Example 37

Find the translation vector which maps the point (6,-6) into (7,16).

Solution

Given that (x, y) = (6,-6) and (x¢, y¢) = (7,16), (a, b) =?

From T (x, y) = (x, y) + (a, b) = (x', y'),

then (7,16) = (6,-6)+(a,b) which means a=7-6 = 1 and b=16+6 = 22. Therefore translation vector ***(a,b) = (1,22).***

The Enlargement Matrix E in Enlarging Figures

Use the enlargement matrix E in enlarging figures

Definition: Enlargement is the transformation which magnifies an object such that its image is proportionally increases on decreased in size by some factor k. The general matrix of enlargement

Is $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ where k is a non-zero real number called ***the linear scale factor***.

So $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

NB: Enlargement is not isometric.

Example 38

Find the image of the square with vertices O(0,0), A (1,0), B (1,1) and C (0,1) under the enlargement matrix $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$. Hence sketch both object and its image on the same axes.

Solution:

Given O = (0,0), A= (1,0), B = (1,1) and C=(0,1) and $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

From

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

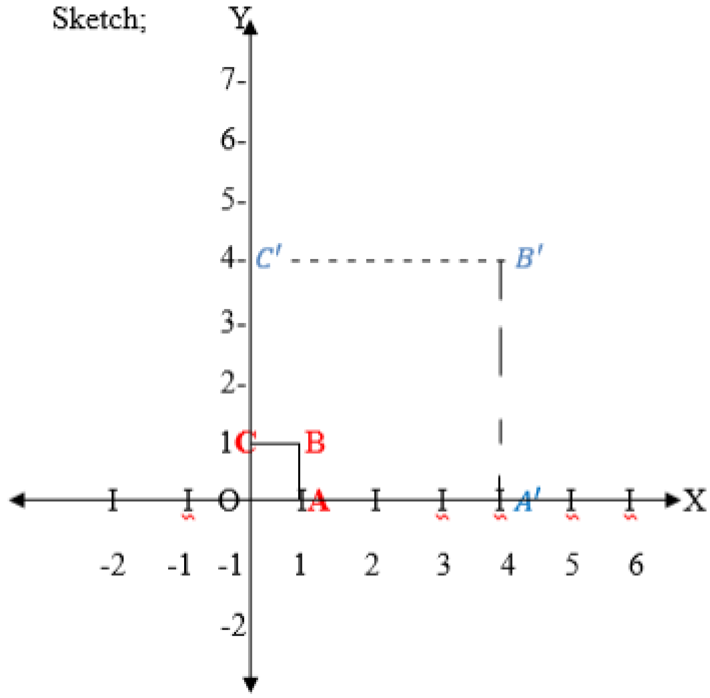
$O' = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $A' = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$,

$B' = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+0 \\ 0+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$,

$C' = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

So $O' = (0,0)$, $A' = (4,0)$, $B' = (4,4)$ and $C' = (0,4)$.

Sketch;



Therefore the image of OABC is OA′B′C′ as shown in the figure above.

Example 39

Find the image of (6, 9) under enlargement by the matrix

$\begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$.

Solution:

From

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

Therefore the image of (6,9) is ***(2,3).***

Example 40

Draw the image of a unit circle with center O (0,0) under

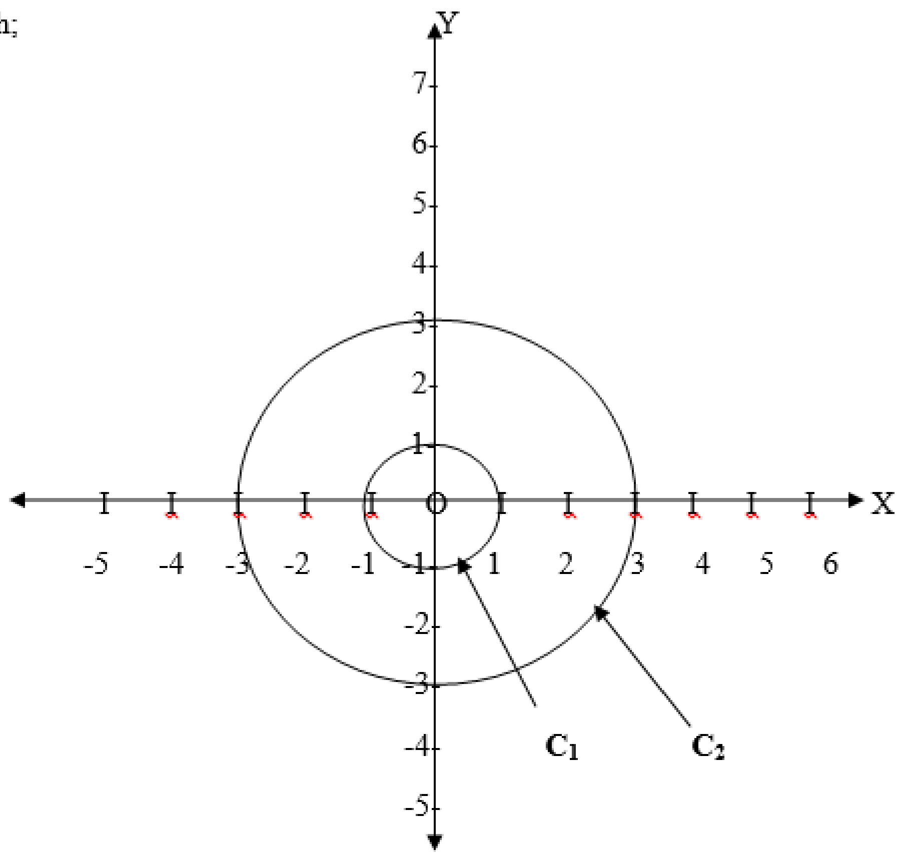
$$M = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution:

Since the circle is centered at (0,0) and has a radius 1 unit, then it passes through the points , (0,1), (1,0) (0,-1), (-1,0) and other points.

Now the images of these points are (0,3), (3,0), (0,-3), (-3,0) and other points respectively, where the centre remains (0,0) and the radius becomes 3 units.

Sketch;



I n the figure above, the circle with radius 1 unit and its image with radius 3 units C_1 and C_2respectively are shown.

Linear Transformation:

Definition:

For any transformation T, any two vectors U and V and any real number t, T is said to be a linear transformation if and only if

T(t U) = tT(U) and ***T (U+V) = T(U) + T(V)***

Example 41

Show that the rotation by 90^0about O(0,0) is a linear trans formation

Solution

Let $U=(U_1,U_2)$ and $V=(V_1, V_2)$ be any two vectors in the plane and t be any real number

To show that $R90^0$ is the linear transformation we must show that

$R90^0\ (tU)= t\ R90^0\ (U)$ and

$R90^0\ (U + V) = R90^0\ (U) + R90^0\ (V)$

$$\text{Now } R90^0\ (U) = \begin{bmatrix} Cos90° & -Sin90° \\ Sin90° & Cos90° \end{bmatrix}\begin{bmatrix} U1 \\ U2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} U1 \\ U2 \end{bmatrix} = \begin{bmatrix} -U2 \\ U1 \end{bmatrix}$$

$$tR90^0\ (U) = t\begin{bmatrix} -U2 \\ U1 \end{bmatrix} = \begin{bmatrix} -tU2 \\ tU1 \end{bmatrix} \text{ while } tU = (tU_1, tU_2) \text{ because } U=(U_1,U_2)$$

$$\text{So } R90^0\ (tU) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} tU1 \\ tU2 \end{bmatrix}$$

$$= \begin{bmatrix} -tU2 \\ tU1 \end{bmatrix}$$

$$\text{Therefore } R90^0\ (tU) = tR90^0\ (U) = \begin{bmatrix} -tU2 \\ tU1 \end{bmatrix} \quad \text{Or } (-tU_2, tU_1)$$

$$\text{Again } R90^0\ (U) = \begin{bmatrix} -U2 \\ U1 \end{bmatrix} \quad \text{and } R90^0\ (V) = \begin{bmatrix} -V2 \\ V1 \end{bmatrix}$$

$$\text{So } R90^0\ (U) + R90^0\ (V) = \begin{bmatrix} -U2 \\ U1 \end{bmatrix} + \begin{bmatrix} -V2 \\ V1 \end{bmatrix}$$

$$= \begin{bmatrix} -(U2 + V2) \\ (U1 + V1) \end{bmatrix}$$

and $U+V = (U_1,U_2)+ (V_1,V_2) = (U_1+V_1),(U_2+V_2) = \big((U_1+V_1),(U_2+V_2)\big)$

$$R90^0\ (U+V) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} (U1 + V1) \\ (U2 + V2) \end{bmatrix} = \begin{bmatrix} -(U2 + V2) \\ (U1 + V1) \end{bmatrix}$$

Therefore, since $R90^0$ (U) + $R90^0$ (V) = $R90^0$ (U+V) and $R90^0$ (tU)= t $R90^0$ (U), then $R90^0$ is a linear trans formation.

Example 42

Suppose that T is a linear transformation such that

T(U) = (1,-2), T(V) = (-3,-1) for any vectors U and V, find

(a) T(U+ V) (b) T(8U) (c) T(3U -2V)

Solution

(a)Since T is a linear Transformation then

T(U+ V) = T(U) + T(V)

$$T(U+V) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

So T(U + V) = (-2 , -3.)

(b) T (8U);

T(8U) = 8 T(U) because T is a linear transformation.

$$\text{So } T(8U) = 8\times\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -16 \end{pmatrix}$$

Therefore T(8U) = (8,-16)

(c) T (3U -2V);

T(3U-2V) = T(3U) – T(2V)

= 3T(U) – 2T(V)

$$=3\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = (9,-4),$$

Therefore T(3U-2V) = (9,-4).

Exercise 7

1. If

$$M=\begin{pmatrix} Cos2\alpha & Sin2\alpha \\ Sin2\alpha & -Cos2\alpha \end{pmatrix}$$

2. Is the matrix of reflection in a line inclined at angle a, U=(6,1) , V=(-1,4) and a13500, find (a) m(U+V) (b) m(2V)

If U =(2,-7) and V=(2,-3), find the matrix of linear transformation T such that T(2U)=(-4,14) and T(3V) = (6,9)

4. What is the image of (1,2) under the transformation

matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ followed by $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$?

5. Given that I is the identify transformation such that I(U) =U for any Vector U, prove that I is a linear transformation.

- READ TOPIC 8: Linear Programming